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The effect of wall conductance on heat diffusion in duct flow

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SUMMARY

The axial diffusion of a passive scalar field (e.g. temperature) in Poiseuille flow through a duct is considered, taking account of leakage of heat through the duct boundary. The cases of the two-dimensional channel and the pipe of circular cross-section are considered in detail, and it is shown that (i) the centroid of the scalar field moves (asymptotically) with a velocity intermediate between the mean and the maximum flow rates and increases with increasing wall conductance, and (ii) the effective diffusivity in the flow direction is a decreasing function of wall conductance.

The temperature field downstream of a maintained heat source is determined as a function of wall conductance.

1. Introduction

The problem of the axial diffusion of a scalar field (e.g. dye concentration or heat) in pipe flow was first considered by Taylor [1] and has been reconsidered from various points of view by a number of subsequent authors (Aris [2], Erdogan and Chatwin [3], Chatwin [4, 5, 6], Dewey and Sullivan [7], Lighthill [8], Smith [9], Sandarasubramanian and Gill [10]). The effective axial diffusion results from a combination of distortion of the scalar field by the mean axial flow (Poiseuille flow in the steady laminar situation) and molecular diffusion, predominantly in the radial direction. When these processes have had a long time to act after the initial release of dye or heat, an asymptotic state is attained in which the centroid of the scalar distribution moves with the mean fluid velocity, and the distribution of the cross-sectional mean about the position of the centroid tends to a Gaussian form controlled by a simple diffusion equation.

In the various treatments cited above, it is assumed that there is zero flux of the scalar field across the pipe boundary, i.e. (in thermal terminology) it is an insulator. This assumption is certainly justified in the case of dye that cannot penetrate the boundary, but in the case of

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heat there is always some leakage through the pipe wall (depending on its thermal conductance), and indeed, if the pipe is metallic, a perfect conductivity condition (rather than an insulating condition) may be appropriate. In many potential applications (e.g. in the problem of heat transport through the cooling circuits of nuclear reactors or, more generally, in any heat exchanger circuit) transfer of heat through the boundary is an essential ingredient of the problem, and will obviously have an important effect either on the decay of temperature in a 'blob' of heat carried through the system, or on the steady temperature distribution established downstream from a maintained source.

In this paper, we consider the influence of finite wall conductance. (We shall use the thermal terminology, although the results may be applicable for other scalar fields as well.) We shall find that, for the convected 'blob' problem, there are three main effects. First, the total heat content, integrated throughout the fluid, is obviously no longer a conserved quantity, since leakage to the walls is now possible. Secondly, the centroid of the temperature distribution in the fluid no longer moves with the mean fluid velocity U_0, but with a velocity U_e intermediate between the mean and the maximum, the precise value depending on the wall conductance and the duct cross-section. Thirdly, the effective axial diffusivity D_e in the asymptotic state is less than that for an insulating boundary (D_0) by a factor which again depends on wall conductance and duct geometry. In the extreme case of a two-dimensional channel with perfectly conducting walls, $U_e = 1.3\,U_0$ and $D_e = 0.14\,D_0$.

These last two effects may both be understood in physical terms as follows: as a patch of hot fluid is distorted by the flow, it is the slower moving parts of the distribution that preferentially diffuse to the wall; the slow tail of the distribution is thus continuously eroded, and so the centroid tends to move faster and the net spreads relative to the centroid tends to be less.

After this paper had been prepared reference [10] came to our notice. Sankarasubramanian and Gill describe the results concerning the centroid and effective diffusivity. The present paper goes rather further than reference [10], inasmuch as it describes the effects of duct geometry on diffusion and extends the work to a study of a temperature field downstream of a maintained heat source. Our approach is simpler, physically more illuminating and may help to achieve a better understanding of the solutions described in [10]. We feel, therefore, that our paper justifies independent publication.

2. Heat diffusion in a duct of arbitrary cross-section

Suppose that fluid flows steadily with velocity $(U(y,z), 0, 0)$ along a duct whose interior is the domain $\mathscr{D}$ in the y–z plane. We consider a temperature field $\theta(\mathbf{x}, t)$ in the fluid which is distorted by the flow and subject to molecular diffusivity κ; the equation for the evolution of $\theta(\mathbf{x}, t)$ is then

$$\frac{\partial\theta}{\partial t} + U(y,z)\frac{\partial\theta}{\partial x} = \kappa\,\nabla^2\theta \quad \text{in } \mathscr{D}\ . \tag{2.1}$$

We suppose further that the thermal boundary condition on the boundary $\partial\mathscr{D}$ is that appropriate to a conducting wall with conductance γ, i.e.

$$\frac{\partial \theta}{\partial n} + \gamma \theta = 0 \quad \text{on} \quad \partial \mathscr{D}, \tag{2.2}$$

where, of course, $\gamma > 0$.

The method adopted by Taylor [1] was in effect to consider the evolution of the cross-sectional average $\theta(x, t)$. The approach works when $\gamma = 0$ essentially because this cross-sectional average automatically satisfies the boundary condition (2.2), viz. $\partial\theta/\partial n = 0$ on $\partial\mathscr{D}$. When $\gamma \neq 0$, the cross-sectional average does not satisfy (2.2), and this leads to difficulties in the detailed application of Taylor's method. The more formal approach described in this section appears to be required. The techniques used here, are related to those that have been employed in papers on associated topics. See, for example, Carrier [11], Philip [12] and Chatwin [13]. Taylor's results (in the case of the pipe of circular cross-section) are of course recovered when $\gamma = 0$. It may be noted that the methods of this section may be adapted to deal with a turbulent diffusivity $\kappa(y, z)$ or with the effect of secondary flow if this is present.

It will be convenient to use dimensionless variables. Let b be a length characteristic of the duct cross-section, and let

$$(\xi, \eta, \zeta) = (x, y, z)/b, \qquad \tau = t\kappa/b^2. \tag{2.3}$$

Let U_m be the maximum value of $|U(y, z)|$ in $\mathscr{D}$ and let $u(\eta, \zeta) = U(y, z)/U_m$. Then (2.1) becomes

$$\frac{\partial \theta}{\partial \tau} + Pu(\eta, \zeta) \frac{\partial \theta}{\partial \xi} = \nabla^2 \theta \quad \text{in } \mathscr{D}\ . \tag{2.4}$$

where now $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + \partial^2/\partial\zeta^2$ and $P = U_m b/\kappa$ (the Peclet number). In most situations of interest, $P \gg 1$; we shall assume that $P \geqslant O(1)$ in the following treatment.

In terms of the Fourier transform of θ, defined by

$$\theta(\xi, \eta, \zeta, \tau) = \int_{-\infty}^{+\infty} \hat{\theta}(k, \eta, \zeta, \tau) e^{ik\xi}\, dk, \tag{2.5}$$

(2.4) becomes

$$\frac{\partial \hat{\theta}}{\partial \tau} + iKu(\eta, \zeta)\hat{\theta} = (\nabla^2 - k^2)\hat{\theta}, \tag{2.6}$$

where $K = Pk$, and (2.2) becomes

$$\frac{\partial \hat{\theta}}{\partial n} + \gamma \hat{\theta} = 0. \tag{2.7}$$

We may seek solutions of the form

$$\hat{\theta} = F(K, \eta, \zeta)e^{-p(K)\tau}, \tag{2.8}$$

where

$$-pF + iKuF = (\nabla^2 - k^2)F \quad \text{in } \mathscr{D}, \tag{2.9}$$

and

$$\frac{\partial F}{\partial n} + \gamma F = 0 \quad \text{on} \quad \partial \mathscr{D}. \tag{2.10}$$

This constitutes an eigenvalue problem for determination of the possible values p_n of p and the corresponding eigenfunctions F_n $(n = 0, 1, 2, \ldots)$.

Standard manipulation of (2.9) and (2.10) provides the result

$$(p - k^2) \int |F|^2 dS = \int |\nabla F|^2 dS + \gamma \int_{\partial\mathfrak{D}} |F|^2 dS + iK \int u |F|^2 dS. \tag{2.11}$$

Note in particular that Re $p > k^2$, and

$$(\text{Im} p) \int |F|^2 dS = K \int u |F|^2 dS. \tag{2.12}$$

The eigenvalues may clearly be ordered so that

$$k^2 < \text{Re} p_0 \leqslant \text{Re} p_1 \leqslant \text{Re} p_2 \leqslant \ldots . \tag{2.13}$$

The general solution of the problem (2.6), (2.7), has the form

$$\hat{\theta} = \sum_n A_n F_n(K, \eta, \zeta) e^{-p_n(K)\tau}, \tag{2.14}$$

where the F_n's may be supposed suitably normalized and the A_n's are then determined by conditions at $\tau = 0$. Provided $A_0 \neq 0$ and Re$p_0 <$ Rep_1, the solution for sufficiently large τ then has the form

$$\hat{\theta} \sim A_0 F_0(K, \eta, \zeta) e^{-p_0(K)\tau}, \tag{2.15}$$

and so, from (2.5),

$$\theta \sim \int_{-\infty}^{+\infty} A_0 F_0(K, \eta, \zeta) e^{-p_0(K)\tau} e^{ik\xi} dk. \tag{2.16}$$

This form of solution leads immediately to evaluation of the effective convection velocity U_e and the effective diffusivity D_e. For it is evident that if $p_0(K)$ can be expressed as a power series in iK, i.e.

$$p_0 = p_{00} + iKp_{01} + (iK)^2 p_{02} + \ldots, \tag{2.17}$$

then θ, as given by (2.16), satisfies a differential equation of the form

$$\frac{\partial\theta}{\partial\tau} = -p_{00}\theta - Pp_{01}\frac{\partial\theta}{\partial\xi} - P^2 p_{02}\frac{\partial^2\theta}{\partial\xi^2} - \ldots, \tag{2.18}$$

and so (returning to dimensional variables)

$$U_e = p_{01}U_m, \qquad D_e = -p_{02}U_m{}^2 b^2/\kappa. \tag{2.19}$$

(It turns out that p_{02} is invariably negative). The leading coefficient p_{00} is the decay rate due to transfer of heat to the boundary.

We are therefore led to seek expansions for $F_0(K, \eta, \zeta)$ and $p_0(K)$ in the form

$$F_0(K,\eta,\zeta) = \sum_{m=0}^{\infty} (iK)^m F_{0m}(\eta,\zeta), p_0(K) = \sum_{m=0}^{\infty} (iK)^m p_{0m}; \tag{2.20}$$

substituting in (2.9) and equating successive powers of K to zero, we obtain

$$(\nabla^2 + p_{00})F_{00} = 0, \tag{2.21}$$

$$(\nabla^2 + p_{00})F_{01} = (u - p_{01})F_{00}, \tag{2.22}$$

$$(\nabla^2 + p_{00})F_{02} = -(p_{02} + P^{-2})F_{00} + (u - p_{01})F_{01}, \tag{2.23}$$

etc.,

and we have also the boundary conditions

$$\left(\frac{\partial}{\partial n} + \gamma\right)F_{0m} = 0 \quad \text{on} \quad \partial\mathscr{D} \qquad (m = 0, 1, 2, \ldots). \tag{2.24}$$

Let us suppose for the moment that the problem for F_{00} and p_{00} is solved. We may suppose that F_{00} is normalized so that

$$\int |F_{00}|^2 dS = 1. \tag{2.25}$$

From (2.21) and (2.24) we then have

$$p_{00} = \int |\nabla F_{00}|^2 dS + \gamma \int_{\partial} |\nabla F_{00}|^2 dS = q^2, \quad \text{say} \tag{2.26}$$

where q is real.

Equation (2.22) together with the boundary condition (2.24) (with $m = 1$) is soluble for F_{01} only if a solvability condition is satisfied. Multiplying equation (2.22) by F_{00}^* and integrating over $\mathscr{D}$, we have

$$p_{01} = \int u |F_{00}|^2 dS. \tag{2.27}$$

It is evident from this that p_{01} is real, as anticipated by the notation of (2.18) above. Similarly, (2.23) together with the boundary condition on F_{02} is soluble only if a similar solvability condition is satisfied: multiplying (2.23) by F_{00}^* and integrating over $\mathscr{D}$, we have

$$(p_{02} + P^{-2}) = \int (u - p_{01}) F_{01} F_{00}^* dS, \tag{2.28}$$

and using (2.22), this may be written

$$p_{02} + P^{-2} = p_{00} \int |F_{01}|^2 dS - \int |\nabla F_{01}|^2 dS - \gamma \int_{\partial \mathscr{D}} |F_{01}|^2 dS = -M(\gamma), \text{ say.} \tag{2.29}$$

In terms of this function, (2.19b) becomes

$$D_e = \kappa + \frac{U_0{}^2 b^2}{\kappa} M(\gamma) \tag{2.30}$$

Equation (2.30) is similar to that obtained by Taylor [1] with the refinement of Aris [2] for pipe flow.

The case $\gamma = 0$ is particularly simple, because then

$$F_{00} = \text{constant} = A^{-1/2}, \tag{2.31}$$

where A is the area of the cross-section, and $p_{00} = 0$. Hence from (2.27),

$$p_{01} = A^{-1} \int u dS = \bar{u}, \tag{2.32}$$

where the overbar represents an average over the cross-section. Hence, from (2.19), the effective convection velocity in this case is

$$U_e = U_0 \bar{u} \tag{2.33}$$

as obtained by Taylor [1]. Moreover $M(\gamma)$ is given in this case (from (2.29)) by

$$M(\gamma) = \int |\nabla F_{01}|^2 dS. \tag{2.34}$$

3. Two-dimensional channel

Let $\mathscr{D}$ be the domain $|\eta| < 1$ and suppose that the flow is steady, laminar, and fully developed, so that

$$u = 1 - \eta^2. \tag{3.1}$$

The problem for F_{00}, p_{00}, is trivial. We find that $p_{00} = q^2$ is the smallest non-negative root $(0 \leqslant q \leqslant \pi/2)$ of the equation

$$q \tan q = \gamma, \tag{3.2}$$

and then $F_{00}(\eta)$, normalised to satisfy (2.25), is given by

$$F_{00}(\eta) = \left(1 + \frac{\sin 2q}{2q}\right) \cos q\eta. \tag{3.3}$$

Note that for an insulating boundary $(\gamma = 0)$ we have $q = 0$ and $F_{00}(\eta) = 2^{-1/2}$, while for a perfectly conducting boundary $(\gamma \to \infty)$, $q = \pi/2$ and $F_{00}(\eta) = \cos(\pi\eta/2)$. For all intermediate values of q, we have $F_{00}(\eta) > 0$, for all η, $|\eta| < 1$.

From (2.27), p_{01} is now given by

$$p_{01} = \int_{-1}^{+1} (1 - \eta^2) \cos^2 q\eta \left(1 + \frac{\sin 2q}{2q}\right)^{-1} d\eta = \frac{8q^3 - 6qc + 3s}{6q^2(2q + s)}, \tag{3.4}$$

where we use the abbreviation

$$c = \cos 2q, \qquad s = \sin 2q.$$

Together with (3.2), this defines p_{01} as a function of the wall conductance γ. Note that

$$p_{01}(0) = \frac{2}{3}, \qquad p_{01}(\infty) = \frac{2}{3} + \frac{2}{\pi^2} = 0.8693. \tag{3.5}$$

The ratio of the corresponding convection speeds, from (2.19), is

$$\frac{(U_e)_{\gamma=\infty}}{(U_e)_{\gamma=0}} = 1 + \frac{3}{\pi^2} = 1.304. \tag{3.6}$$

As anticipated in the introduction, the effective convection speed is greater when the walls are conducting than when they are insulating.

The function $F_{01}(\eta)$ may now be obtained by integrating (2.22) subject to the symmetry condition

$$F'_{01}(0) = 0. \tag{3.7}$$

The result is

$$\left(1 + \frac{\sin 2q}{2q}\right)^{1/2} F_{01}(\eta) = -\frac{\eta^2}{4q^2}\cos q\eta + \left(g(q)\eta - \frac{\eta^3}{6q}\right)\sin q\eta + BF_{00}(\eta) \tag{3.8}$$

where B is an undetermined constant, and

$$g(q) = \frac{2q^2 + 3c + 3qs + 3}{6q^2(2q + s)}. \tag{3.9}$$

By virtue of (3.4), it may be verified that the function (3.8) automatically satisfies the conditions

$$\frac{dF_{01}}{d\eta} + \gamma F_{01} = 0 \quad \text{on} \quad \eta = \pm 1. \tag{3.10}$$

Finally, p_{02} may be obtained from (2.28); note that, by virtue of (2.27), the term involving the constant B in (3.8) makes zero contribution to the integral. We obtain the following result

$$p_{02} + P^{-2} = \left(1 + \frac{s}{2q}\right)^{-1} [J(1 - p_{01})^2 + K(1 - p_{01}) + L] = -M(\gamma), \tag{3.11}$$

where

$$J = \tfrac{1}{8} q^{-2}s - \tfrac{1}{4} q^{-1}c,$$

$$K = \tfrac{1}{12} q^{-2} + (\tfrac{3}{8} q^{-5} - \tfrac{5}{8} q^{-3})s + (-\tfrac{1}{2} q^{-4} + \tfrac{1}{3} q^{-2})c,$$

$$L = \tfrac{1}{20} q^{-2} + (\tfrac{1}{3} q^{-3} - \tfrac{19}{16} q^{-5} + \tfrac{19}{32} q^{-7})s + (-\tfrac{1}{12} q^{-2} + \tfrac{19}{24} q^{-4} - \tfrac{35}{32} q^{-6})c.$$

The extreme cases of insulating and perfectly conducting boundaries are again of particular interest. The former case may be obtained from consideration of the limit $q \to 0$ in (3.9) and (3.11). It is however simpler to solve the problem with $q = 0$ from the outset. This procedure gives

$$F_{00} = 2^{-1/2}, \qquad F_{01} = 2^{-1/2}(\tfrac{1}{6}\eta^2 - \tfrac{1}{12}\eta^4 + B), \tag{3.12}$$

and

$$p_{00} = 0, \qquad p_{01} = \tfrac{2}{3}, \qquad M(0) = \tfrac{8}{945} = 0.00847. \tag{3.13}$$

In the case of perfectly conducting boundaries ($\gamma \to \infty$, $q = \pi/2$), (3.9) and (3.11) yield

$$M(\infty) = \tfrac{4}{45}\pi^{-6}[75\pi^2 - \pi^4 - 630] = 0.00118. \tag{3.14}$$

The ratio

$$\frac{M(\infty)}{M(0)} = 0.140 \tag{3.15}$$

gives a measure of the reduction in effective diffusivity in replacing insulating boundaries by perfect conductors.

4. Pipe of circular cross-section

Suppose now that $\mathscr{D}$ is the domain $r^2 = \eta^2 + \zeta^2 < 1$, and that the flow is again steady, laminar and fully developed, so that

$$u = 1 - r^2. \tag{4.1}$$

The problem for F_{00}, p_{00} is again trivial. From (2.21) and (2.24), together with the condition of finiteness at $r = 0$, we have

$$F_{00}(r) = CJ_0(qr), \qquad p_{00} = q^2, \tag{4.2}$$

where q is the smallest non-negative root of the equation

$$\gamma = -\frac{qJ_0'(q)}{J_0(q)} = \frac{qJ_1(q)}{J_0(q)}, \tag{4.3}$$

and C is given by (2.25), i.e.

$$C = \left\{\int_0^1 J_0{}^2(qr)\, 2\pi r dr\right\}^{-1/2}. \tag{4.4}$$

When $\gamma = 0$, $qJ_1(q) = 0$, and the relevant root is again $q = 0$. When $\gamma = \infty$, $J_0(q) = 0$, the smallest root being $q = 2.405$. As γ increases from 0 to ∞, the smallest positive root of (4.3) increases monotonically from 0 to 2.405, and $0 < F_{00}(r) < C$ for $0 \leqslant r < 1$.
From (2.27), p_{01} is given by

$$p_{01} = \int_0^1 (1 - r^2) J_0{}^2(qr) r\, dr \int_0^1 J_0{}^2(qr) r\, dr. \tag{4.5}$$

Note in particular the extreme values of $p_{01} = 0.5$ when $\gamma = 0$, $p_{01} = 0.78$ when $\gamma = \infty$. The value for $\gamma = 0$ is then as obtained by Taylor [1], while for $\gamma = \infty$ (4.5) gives the result obtained by the authors [10]. The effective convection velocity is therefore a factor ~ 1.6 greater when the pipe is a perfect conductor than when it is an insulator.

The function $F_{01}(r)$ satisfies

$$\frac{1}{r}\frac{d}{dr} r \frac{dF_{01}}{dr} + q^2 F_{01} = C(1 - r^2 - p_{01}) J_0(qr) \tag{4.6}$$

and we require the solution that is finite at $r = 0$; by virtue of (4.5), this solution then automatically satisfies the condition

$$\left(\frac{\partial}{\partial r} + \gamma \ \ F_{01}\right) = 0 \quad \text{on} \quad r = 1. \tag{4.7}$$

The required solution is

$$F_{01}(r) = \frac{\pi}{2}\int_0^r [J_0(qr')Y_0(qr) - J_0(qr)Y(qr')]\, C(1 - r'^2 - p_{01}) J_0(qr')\, r' dr'. \tag{4.8}$$

The function $M(\gamma) = -(p_{02} + P^{-2})$ may now be computed from (2.29). It is found that $M(\gamma)$ decreases monotonically from 0.0052 to 0.00124 as γ increases from 0 to ∞. The value for $\gamma = 0$ is then as obtained by Taylor [1] and Aris [8].

In this case

$$\frac{M(\infty)}{M(0)} = 0.25, \tag{4.9}$$

a result that may be compared with the two-dimensional result (3.15). The variation of p_{00}, p_{01} and $p_{02} + P^{-2}$ as functions of γ for pipe flow are shown by the dotted curves in Figure 1.

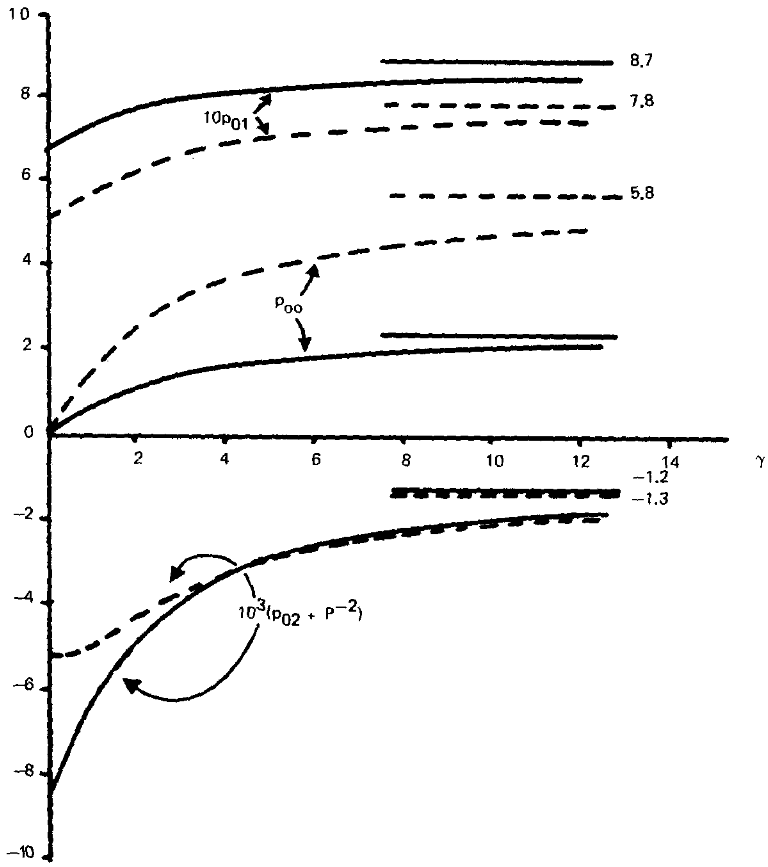

Figure 1. Variation of p_{00}, p_{01} and $p_{02} + P^{-2}$ as functions of γ.
——— Two-dimensional channel;
– – – – Pipe of circular cross-section.

5. The behaviour of higher harmonics

If we write the Fourier inverse of (2.14) in the form

$$\theta(\xi, \eta, \zeta, \tau) = \sum_{n=0}^{\infty} \theta_n(\xi, \eta, \zeta, \tau), \tag{5.1}$$

then the analysis of Section 2 has shown that the leading term satisfies the modified diffusion equation

$$\frac{\partial \theta_0}{\partial \tau} + P p_{01} \frac{\partial \theta_0}{\partial \xi} = -p_{00}\theta_0 - P^2 p_{02} \frac{\partial^2 \theta_0}{\partial \xi^2} + \dots, \tag{5.2}$$

(with $p_{02} < 0$). Moreover, the structure of this leading term is determined by the function $F_{00}(\eta, \zeta)$; in fact as $\tau \to \infty$

$$\theta_0(\xi, \eta, \zeta, \tau) \sim A_0 F_{00}(\eta, \zeta) \frac{e^{-p_{00}\tau}}{(-P^2 p_{02}\tau)^{1/2}} \exp \frac{-(\xi - P p_{01}\tau)^2}{(-4P^2 p_{02}\tau)}, \tag{5.3}$$

is a Gaussian function centred on $\xi = P p_{01}\tau$.

A similar analysis may be applied to the terms θ_n for $n \geqslant 1$. The functions $F_n(K, \eta, \zeta)$, $p_n(K)$, determined by the eigenvalue problem (2.9), (2.10) may be expanded (cf. (2.20)) in the form

$$F_n(K, \eta, \zeta) = \sum_{m=0}^{\infty} (iK)^m F_{nm}(\eta, \zeta), \qquad p_n(K) = \sum_{m=0}^{\infty} (iK)^m p_{nm}, \tag{5.4}$$

and p_{nm}, $F_{nm}(\eta, \zeta)$ may then be determined by perturbation procedure. It is evident that this will lead to an effective velocity $U_e^{(n)} = p_{n1} U_0$ and an effective diffusivity $D_e^{(n)} = -p_{n2} U_0^2 b^2/\kappa$ $(p_{n2} < 0)$ for each harmonic $n = 1, 2, 3, \ldots$, and that for $\tau \to \infty$, (cf. (5.3)),

$$\theta_n(\xi, \eta, \xi, \tau) \sim A_n F_{n0}(\eta, \zeta) \frac{e^{-p_{n0}\tau}}{(-P^2 p_{n2}\tau)^{1/2}} \exp \frac{-(\xi - P p_{n1}\tau)^2}{(-4P^2 p_{n0}\tau)}. \tag{5.5}$$

The fact that the different ingredients θ_n move with different effective velocities $p_{n1} U_0$ suggests that there may be a detectable separation of the ingredients as $\tau \to \infty$. However, of the structure functions $F_{n0}(\eta, \zeta)$, only $F_{00}(\eta, \zeta)$ is positive over the whole cross-section $\mathscr{D}$, and it is clear, on physical grounds, that if $\theta(\mathbf{x}, 0) \geqslant 0$ for all $\mathbf{x}$, then $\theta(\mathbf{x}, t) \geqslant 0$ for all $\mathbf{x}$ and all $t > 0$. It follows that the leading term of (5.1) must in fact dominate the series for all ξ, despite the differences in effective convection velocities of the different ingredients.

This consequence of the positivity of θ has interesting implications concerning the relative magnitude of the constants p_{nm}. Suppose we move with the convection velocity of θ_n $(n \geqslant 1)$, i.e. $\xi = P p_{n1}\tau$; then

$$\theta_n \propto \tau^{-1/2} e^{-p_{n0}\tau} \tag{5.6}$$

and

$$\theta_0 \propto \tau^{-1/2} e^{-p_{00}\tau} \exp \frac{-(p_{n1} - p_{01})^2 \tau}{-4p_{02}}, \tag{5.7}$$

and the requirement that θ_0 dominate over θ_n for all τ implies that

$$p_{n0} \geqslant p_{00} - \frac{(p_{n1} - p_{01})^2}{4p_{02}} \qquad (n = 1, 2, 3, \ldots). \tag{5.8}$$

Under this condition, the θ_n-ingredient has a natural decay rate p_{n0} which more than compensates for the fact that it finds itself further and further out in the tail of the θ_0-Gaussian distribution.

The inequality (5.8) is well illustrated by the simplest case of a two-dimensional channel. For the first harmonic

$$F_{10}(\eta) = \left(1 - \frac{\sin 2q}{2q}\right)^{-1/2} \sin q\eta \qquad (5.9)$$

Note that for an insulating boundary ($\gamma = 0$) we have $q = \pi/2$, $P_{10}(0) = \pi^2/4 = 2.4674$, $F_{10}(\eta) = \sin(\pi\eta/2)$, while for a perfectly conducting boundary ($\gamma \to \infty$), $q = \pi$, $F_{10}(\eta) = \sin \pi\eta$, and $P_{10}(\infty) = \pi^2$.

From (2.27), P_{11} is now given by

$$P_{11} = \left(1 - \frac{\sin 2q}{2q}\right)^{-1} \int_{-1}^{+1} (1-\eta^2)\sin^2 q\eta\, d\eta = \frac{8q^3 - 3S + 6qC}{6q^2(2q - S)}, \qquad (5.10)$$

where $C = \cos 2q$ and $S = \sin 2q$. Note that

$$P_{11}(0) = \frac{2}{3} - \frac{2}{\pi^2} = 0.4640, \qquad P_{11}(\infty) = \frac{2}{3} + \frac{1}{2\pi^2} = 0.7173. \qquad (5.11)$$

$P_{12} + P^{-2}$ is now given by

$$P_{12} + P^{-2} = (1 - \tfrac{1}{2} Sq^{-1})^{-1}\{(1 - P_{11})^2 R + (1 - P_{11})T + W\} = M_1(\gamma), \qquad (5.12)$$

where

$$\begin{aligned} R &= \tfrac{1}{4} Cq^{-2} - \tfrac{1}{8} Sq^{-3}, \\ T &= \tfrac{1}{12} q^{-2} + [\tfrac{5}{8} q^{-3} - \tfrac{3}{8} q^{-5}] S + [\tfrac{3}{4} q^{-4} - \tfrac{1}{3} q^{-2}] C, \\ W &= \tfrac{1}{20} q^{-2} + [-\tfrac{1}{3} q^{-3} + \tfrac{19}{16} q^{-5} - \tfrac{19}{32} q^{-7}] S \\ &\quad + [\tfrac{1}{12} q^{-2} - \tfrac{19}{24} q^{-4} + \tfrac{19}{16} q^{-6}] C. \end{aligned} \qquad (5.13)$$

The extreme cases of insulating and perfectly conducting boundaries are again of particular interest. When $\gamma = 0$, $M(0) = -0.003355$ and when $\gamma = \infty$, $M(\infty) = -0.001116$.

From Section 3, $P_{00} = 0$, $P_{01} = \tfrac{2}{3}$, $P_{02} = -0.00847$. Hence,

$$P_{00} - \frac{(P_{11} - P_{01})^2}{4P_{02}} = 0.21235 \qquad (5.14)$$

(when $\gamma = 0$) and the inequality (5.8) is satisfied for $n = 1$, although not by a wide margin. When $\gamma = \infty$

$$P_{00} - \frac{(P_{11} - P_{01})^2}{4P_{02}} = 7.3623 \tag{5.15}$$

Again the inequality (5.8) is satisfied.

6. Downstream diffusion from a maintained source

If the cross-sectional distribution of θ is maintained at some section, $\xi = 0$ say, then a steady temperature distribution will be established downstream, i.e. for $\xi > 0$. This steady distribution has the form

$$\theta \sim \sum_{n=0}^{\infty} A_n F_{n0}(\eta, \zeta)\, \theta_n(\xi) \quad \text{as} \quad \xi \to \infty, \tag{6.1}$$

where, from (5.2) and similar equations for θ_n,

$$-P^2 p_{n2} \frac{d^2\theta_n}{d\xi^2} - P p_{n1} \frac{d\theta_n}{d\xi} - p_{n0}\theta_n = 0, \qquad (n = 0, 1, 2, \ldots) \tag{6.2}$$

These equations have solutions of the form

$$\theta_n(\xi) = e^{-\alpha_n \xi}, \tag{6.3}$$

where

$$\alpha_n = \frac{1}{2P}\left\{\left[\left(\frac{p_{n1}}{p_{n2}}\right)^2 - 4\frac{p_{n0}}{p_{n2}}\right]^{1/2} + \frac{p_{n1}}{p_{n2}}\right\} > 0. \tag{6.4}$$

Consider first the case of an insulating boundary $\gamma = 0$. In this case $p_{00} = 0$ and $p_{n0} > 0$ $(n \geqslant 1)$. Hence $\alpha_0 = 0$ and $\alpha_n > 0 (n \geqslant 1)$. As is physically obvious, the temperature distribution in this case tends to the constant value $A_0 F_{00}$ as $\xi \to \infty$, and the length of duct over which the temperature is significantly non-uniform is in general of the order of

$$L_1 = \alpha_1^{-1} b = 2Pb\left\{\left[\left(\frac{p_{11}}{p_{12}}\right)^2 - 4\frac{p_{10}}{p_{12}}\right]^{1/2} + \frac{p_{11}}{p_{12}}\right\}^{-1}. \tag{6.5}$$

When $\gamma \to 0$ (for a two-dimensional case)

$$\frac{L_1}{Pb} = \frac{2(\pi^3 - 3\pi)}{3\pi^3\gamma}, \tag{6.6}$$

where (5.11) and the relevant expressions for p_{12} and p_{10} have been used. The quantity in curly brackets in (6.5) depends on the duct cross-section, but is invariably a number of order units.

If the boundary is not insulating (i.e. $\gamma > o$), then $\alpha_0 > 0$, and

$$\theta \sim A_0 F_{00}(\eta, \zeta)\, e^{-\alpha_0 \xi} \quad \text{as} \quad \xi \to \infty. \tag{6.7}$$

The temperature field decays over a distance

$$L_0 = \alpha_0^{-1} b = 2Pb \left\{ \left[\left(\frac{p_{01}}{p_{02}}\right)^2 - 4\frac{p_{00}}{p_{02}} \right]^{1/2} + \frac{p_{01}}{p_{02}} \right\}^{-1} . \tag{6.8}$$

Here the quantity in curly brackets depends on the duct cross-section and on γ. The asymptotic results for small and large γ are as indicated in the following table for the case of the two-dimensional channel (Section 3) and the pipe of circular cross-section (Section 4).

Table 1. Behaviour of L_0/Pb for small and large values of the wall conductance γ.

	$\gamma \to o$	$\gamma \to \infty$
Two-dimensional channel	$\sim 2/3\gamma$	0.3537
Pipe of circular cross-section	$\sim 1/2\gamma$	0.1365

7. Discussion

It has been shown that if the duct is a thermal conductor, then in addition to the net loss of heat from the fluid to the boundary, the values of the effective convection velocity U_e and the effective diffusivity D_e are altered; in fact U_e turns out to be an increasing function of the wall conductance, while D_e is a decreasing function of the conductance. This information should have a direct effect on the interpretation of data in several heat and mass transfer systems, notably in flow systems with heterogeneous catalysis. The results are consistent with earlier results of Sankarasubramanian and Gill [10], but the method adopted in the present paper is simpler and therefore more easily generalisable.

The continuous loss of heat from the fluid in the case of a conducting boundary means that the effect of a maintained source of heat at a given section will penetrate only a finite distance downstream. This distance is calculated, as a function of wall conductance, in the final section of this paper. This result is of practical importance in many branches of engineering. The example of cooling circuits in nuclear reactors, mentioned in the introduction, is one possible application. Another potential application is in screw extruders where knowledge of the distance the heat penetrates is essential. This paper has shed some light on the order of magnitude of this distance.

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